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"Study of Modern Instrumentation and Methods for Astronomic Positioning in the Field"

SECOND TECHNICAL REPORT

Stuttgart, March 1988

Summary

The *First Technical Report* contained a detailed simulation study about the possible accuracy of *astronomic positioning* in the field. Namely we refer to Chapter Three, especially to *Figures* 2-29. The simulation study was based on the assumption that the objects of field observations, the stars, are perfectly known, including also the various astronomic reductions such as proper motion, aberration parallax etc. Here we, therefore, allow some uncertainty of the star position: We present a detailed analysis of how to proceed once only *dispersive prior information* about star positions is available. The nonlinear condition equations are linearized by a two-step Taylor series expansion accounting for stochastic information $\hat{\alpha}$, $\hat{\delta}$ of right ascension and declination, the parameters of star positions. A numerical analysis will be presented in the *Final Report*.

0. Introduction

Within the *First Technical Report* (March 1987) we have characterized astronomic positioning as the determination of the direction parameters of the gravity vector, namely of *astronomical longitude* Λ and *astronomical latitude* Φ, relative to an earth-fixed reference frame. According to the observational techniques of stellar objects by means of a theodolite, an additional quantity, the *orientation unknown* Σ *in the horizontal plane* has to be determined. Within a *Gauß-Helmert model* of condition equations with unknowns we simulated various observational configurations in order to get an accuracy estimate of the field operations by which astronomical longitude Λ, latitude Φ and the orientation unknown Σ are determined. Namely we refer to Chapter Three of the *First Technical Report* (March 1987), especially to *Figures 2-29*. The accuracy of the observations had been assumed as follows: The r.m.s. value for the *horizontal direction* was stated as σ_T = 1", for the *vertical direction* σ_B = 1" and for *time measurement* σ_θ = 0.1 sec.

The simulation study was performed under the restriction that the respective star positions are given *perfectly*. In reality, the star positions, namely right ascension α and declination δ, have to be characterized by a variance-covariance matrix. Its influence on astronomic positioning is analyzed herein. We derive observational equations in which the uncertainty of the star position is represented. Only through such a general model can the accuracy of *astronomic positioning in the field* be realisticallly estimated.

1. Relations between the reference frames in geodetic astronomy

First of all in this chapter the fundamental relations in geodetic astronomy between observations, unknowns and given coordinates shall be derived which will be needed further.

1.1 The systematical structure of the reference frames

The reference frames used in geodetic astronomy may be arranged on different levels which are numbered in turn or indicated by symbols: 0 corresponds to ', 1 to *, 2 to •, 3 to ○. One fundamental vector $\underset{\sim}{V}^i$ belongs to every level i. In details this is as follows:

$\underset{\sim}{V}^0 = \underset{\sim}{V}' = \underset{\sim}{Z}$	the position vector from the point of observation to the target object (terrestrial or celestial), which is generally a star;
$\underset{\sim}{V}^1 = \underset{\sim}{V}^* = -\underset{\sim}{\Gamma}$	the negative gravity vector;
$\underset{\sim}{V}^2 = \underset{\sim}{V}^{\bullet} = \underset{\sim}{\Omega}$	the earth rotation vector (it has the direction of the axis of the earth, points to the North Pole and has the value of the earth rotation rate);
$\underset{\sim}{V}^3 = \underset{\sim}{V}^{\circ} = \underset{\sim}{\psi}$	the ecliptic normal vector (it points to the northern pole of the ecliptic).

An orthonormal reference frame $\underset{\approx}{E}^i$ belongs to every level with its base vectors as follows:

$$\underset{\sim}{E}^i_3 = \text{norm } \underset{\sim}{V}^i \tag{1-1}$$

$$\underset{\sim}{E}^i_2 = \text{norm } (\underset{\sim}{V}^{i+1} \times \underset{\sim}{V}^i) \tag{1-2}$$

$$\underset{\sim}{E}^i_1 = \underset{\sim}{E}^i_2 \times \underset{\sim}{E}^i_3 \tag{1-3}$$

Here "norm" denotes the abbreviation for the normalization of a vector, and "x" the vector product.

New reference frames are at the lower end the observational frame $\underset{\approx}{E}'$ of the level

"0", whose third base vector is located in the direction of the observation and which is unique since it is not a reference frame in the literal sense, because there are no vectors described with regard to this frame, and at the upper end the ecliptic frame $\underset{\simeq}{E}^{o}$, which has hardly any practical importance.

In addition to the systematical $\underset{\simeq}{E}$-triads, $\underset{\simeq}{F}$-triads also appear on each level. These systems have the common third base vector with the appertaining $\underset{\simeq}{E}$-frame, nevertheless the direction of the first and second base vector does not follow from the systematic structure of the fundamental vectors $\underset{\sim}{V}^{i}$, $\underset{\sim}{V}^{i+1}$, but from a more or less arbitrary definition.

A new $\underset{\simeq}{F}$-triad is the theodolite frame $\underset{\simeq}{F}^{*}$, whose first base vector $\underset{\sim}{F}_{1*}$ lies in the direction "zero" of the azimuth circle of a theodolite which is set up in the astronomical horizon. The longitudinal angle (see below) of an observed direction in the local horizon frame is the horizontal direction T and is recorded systematically *in* the clockwise direction, but conventionally *counter*-clockwise, $T_s = -T_c$. The latitude angle (see below) is the vertical direction as in the horizon frame $\underset{\simeq}{E}^{*}$.

The transformation from a frame $\underset{\simeq}{E}^{i}$ to the appertaining frame $\underset{\simeq}{F}^{i}$ is always a counter-clockwise rotation round the common third axis with the orientation angle H^{i},

$$\underset{\simeq}{F}^{i} = \underset{=}{R}_3(H^{i})\underset{\simeq}{E}^{i} \; . \tag{1-4}$$

$\underset{=}{R}_3$ is the rotation matrix, which describes a rotation of a frame round the third axis. It is

$$\underset{=}{R}_3(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1-5}$$

Corresponding to eqn. (1-5) the rotation matrices for the rotations round the first and second axis read

$$\underline{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \tag{1-6}$$

$$\underline{R}_2(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \tag{1-7}$$

$\underline{R}_1$, $\underline{R}_2$ and $\underline{R}_3$ are also called elementary rotations. The orientation angles H^i (see eqn. (1-4)) are in detail:

$H^1 = H^* = \Sigma$ — the orientation unknown of the theodolite which has been set up;

$H^2 = H^{\bullet} = \theta_{Gr,s}$ — the Greenwich sidereal time;

$H^3 = H^{\circ}$ — the angle between the line of intersection of the ecliptic with the mean galaxy plane and the direction to the vernal equinox $\pm 90°$.

For the transformation from a frame $\underline{E}^{i+1}$ to the underlying frame $\underline{E}^i$ one needs the longitudinal angle χ^i_{i+1} and the latitude angle Φ^i_{i+1} of the fundamental vector $\underset{\sim}{V}^i$ with regard to the frame $\underline{E}^{i+1}$:

$$\begin{aligned} \underline{E}^i = \underline{R}_2(90°-\Phi^i_{i+1})\ \underline{R}_3(\chi^i_{i+1})\ \underline{E}^{i+1} &= \underline{R}_E(\chi^i_{i+1},\Phi^i_{i+1},0)\underline{E}^{i+1} \\ &= \underline{R}_E(\chi^i_{i+1},\Phi^i_{i+1})\underline{E}^{i+1} \quad . \end{aligned} \tag{1-8}$$

$\underline{R}_E$ is the special case of a rotation matrix of Eulerian type, in which the three elementary rotations are connected in a row as follows:

- first rotation round the third axis, $\underline{R}_3(\gamma_1)$
- second rotation round the new second axis with the angle $(90°-\beta)$, $\underline{R}_2(90°-\beta)$
- third rotation round the new third axis

ble 1: The $\underset{\approx}{E}$ and $\underset{\approx}{F}$ reference frames

level	symbol	fundamental vector	Notation and name of the $\underset{\approx}{E}$-frame	3. base vector of the $\underset{\approx}{E}$-frame	2. base vector of the $\underset{\approx}{E}$-frame	1. base vector of the $\underset{\approx}{E}$-frame	Notation and name of the $\underset{\approx}{F}$-frame	1. base vector of the $\underset{\approx}{F}$-frame
0	'	$\underset{\sim}{Z}$	$\underset{\approx}{E}'$ observational frame	$\underset{\sim}{E}_{3'} = \text{norm } \underset{\sim}{Z}$ observational direction	$\underset{\sim}{E}_{2'} = \text{norm}(-\underset{\sim}{\Gamma} x \underset{\sim}{Z})$ (in the horizontal plane)	$\underset{\sim}{E}_{1'} = \underset{\sim}{E}_{2'} x \underset{\sim}{E}_{3'}$		
1	*	$-\underset{\sim}{\Gamma}$	$\underset{\approx}{E}^{*}$ horizontal frame	$\underset{\sim}{E}_{3*} = \text{norm}(-\underset{\sim}{\Gamma})$ zenith	$\underset{\sim}{E}_{2*} = \text{norm}(\underset{\sim}{\Omega} x (-\underset{\sim}{\Gamma}))$ east	$\underset{\sim}{E}_{1*} = \underset{\sim}{E}_{2*} x \underset{\sim}{E}_{3*}$ south	$\underset{\approx}{F}^{*}$ theodolite frame	$\underset{\sim}{F}_{1*}$, direction "zero" of the azimuth circle of the theodolite
	•	$\underset{\sim}{\Omega}$	$\underset{\approx}{E}^{\bullet}$ equatorial frame (fixed in space)	$\underset{\sim}{E}_{3\bullet} = \text{norm}\underset{\sim}{\Omega}$ north pole	$\underset{\sim}{E}_{2\bullet} = \text{norm}(\underset{\sim}{\psi} x \underset{\sim}{\Omega})$ autumn equinox	$\underset{\sim}{E}_{1\bullet} = \underset{\sim}{E}_{2\bullet} x \underset{\sim}{E}_{3\bullet}$	$\underset{\approx}{F}^{\bullet}$ equatorial frame (fixed in the earth)	$\underset{\sim}{F}_{1\bullet}$, in the Greenwich meridian
	°	$\underset{\sim}{\psi}$	$\underset{\approx}{E}^{\circ}$ ecliptical frame (systematical)	$\underset{\sim}{E}_{3\circ} = \text{norm}\underset{\sim}{\psi}$ northern pole of the ecliptic	$\underset{\sim}{E}_{2\circ} = \text{norm}(\underset{\sim}{V}^{4} x \underset{\sim}{\psi})$ (in the average galaxy)	$\underset{\sim}{E}_{1\circ} = \underset{\sim}{E}_{2\circ} x \underset{\sim}{E}_{3\circ}$	$\underset{\approx}{F}^{\circ}$ ecliptical frame (conventional)	$\underset{\sim}{F}_{1\circ}$, vernal equinox
			$\underset{\approx}{E}^{4}$ average galaxy frame	$\underset{\sim}{E}_{3^4} = \text{norm}\underset{\sim}{V}^{4}$	$\underset{\sim}{E}_{2^4} = \text{norm}(\underset{\sim}{V}^{5} x \underset{\sim}{V}^{4})$	$\underset{\sim}{E}_{1^4} = \underset{\sim}{E}_{2^4} x \underset{\sim}{E}_{3^4}$		

...

In the matrix $\underset{-}{R}_e$ the second rotation round the second axis would take place with the angle $\beta, \underset{-}{R}_2(\beta)$.

These longitudinal and latitude angles are in detail:

$\chi_1^0 = \chi'_* = A_s$ the azimuth of the observational direction;

$\Phi_1^0 = \Phi'_* = B$ the vertical direction

$\chi_2^1 = \chi^*_\bullet = \theta_s$ the sidereal time

$\phi_2^1 = \phi^*_\bullet = \phi$ the astronomical latitude.

For the transformation from a frame $\underset{\simeq}{E}^{i+1}$ to the frame F^i lying diagonally underneath, one needs additionally the orientation angle $H^{\tilde{i}}$ (see Fig. 1):

$$\underset{\simeq}{F}^i = \underset{-}{R}_E(\chi^i_{i+1}, \phi^i_{i+1}, H^i)\underset{\simeq}{E}^{i+1} \tag{1-9}$$

For the transformation from a frame $\underset{\simeq}{F}^{i+1}$ to the frame $\underset{\simeq}{E}^i$ lying diagonally underneath, one needs the longitudinal angle Λ^i_{i+1} and the latitude angle Φ^i_{i+1} of the fundamental vector $\underset{\sim}{V}^i$ with regard to the frame $\underset{\simeq}{F}^{i+1}$:

$$\underset{\simeq}{E}^i = \underset{-}{R}_E(\Lambda^i_{i+1}, \phi^i_{i+1})\underset{\simeq}{F}^{i+1} \tag{1-10}$$

The latitude angles are the same as above, the longitudinal angles are in detail:

$\Lambda_1^0 = \Lambda'_* = T_s$ the horizontal direction of the observation direction, systematically measured counter-clockwise, conventionally in the clockwise direction, $T_s = -T_c$;

$\Lambda_2^1 = \Lambda^*_\bullet = \Lambda$ the astronomical longitude;

$\Lambda_3^2 = \Lambda^\bullet_\circ = 90°$

and

$\phi_3^2 = \phi^\bullet_\circ = 90° - \varepsilon$ the orthogonal complement to the inclination of the ecliptic.

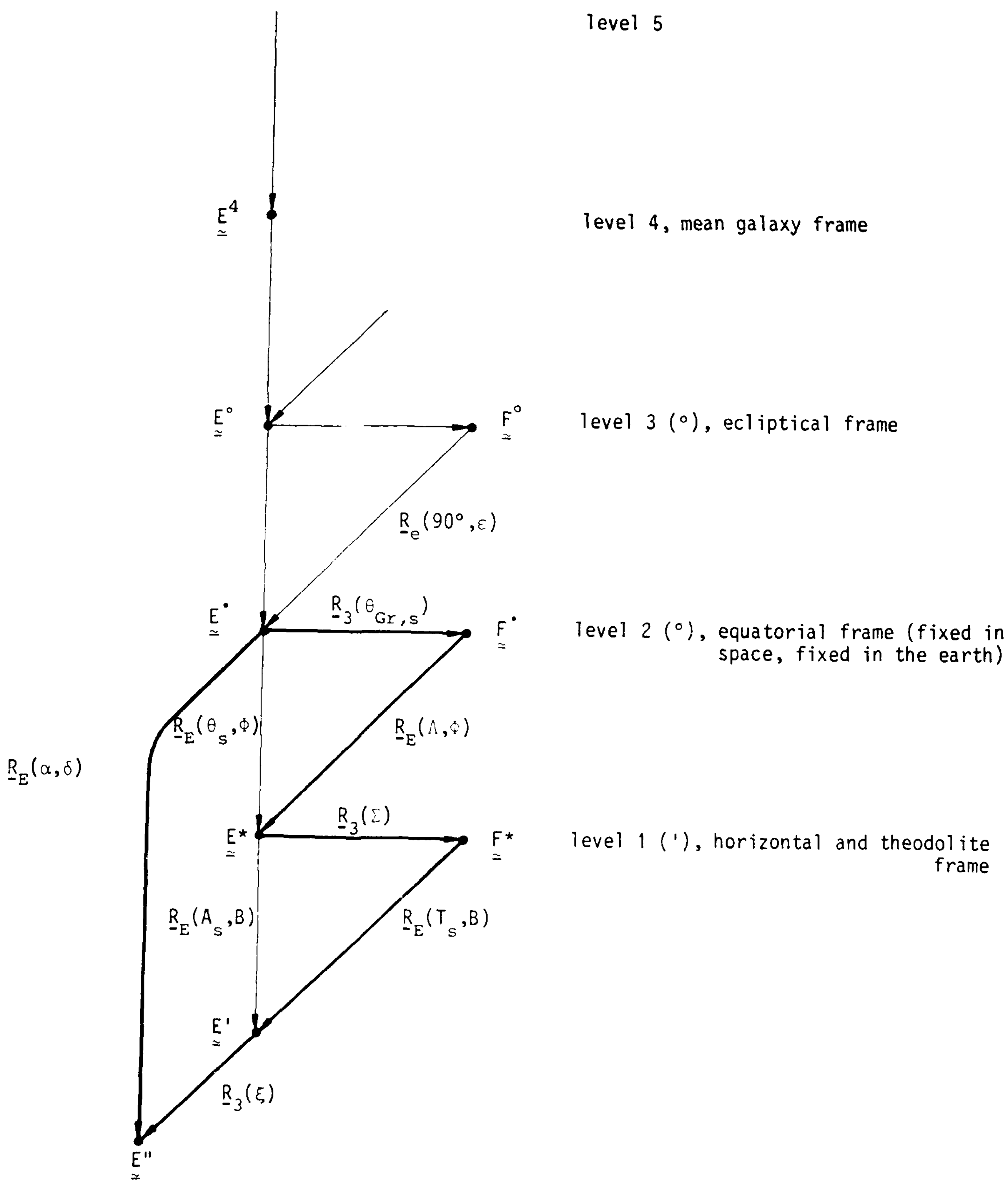

Fig. 1 : Commutative diagram with reference frames in geodetic astronomy

For the transformation from a frame $\underset{\simeq}{F}^{i+1}$ to the underlying frame $\underset{\simeq}{F}^{i}$, one needs additionally the orientation angle H^i:

$$\underset{\simeq}{F}^{i} = \underline{R}_E(\Lambda^i_{i+1},\Phi^i_{i+1},H^i)\underset{\simeq}{F}^{i+1} \tag{1-11}$$

The transformation in the opposite direction results from the respective transposed of the rotation matrix.

1.2 The commutative diagram and the fundamental equation of geodetic astronomy

In the rotations introduced up to this point the right ascension α and the declination δ are still missing; these describe the observation direction $\underset{\sim}{E}_{3'}$ to a star with regard to the space fixed equator frame, namely in the same way as A and B do this with regard to the horizontal frame. Indeed, the rotation $\underline{R}_E(\alpha,\delta)\underset{\simeq}{E}^{\bullet}$ does not lead to the frame $\underset{\simeq}{E}'$; in $\underset{\simeq}{E}'' = \underline{R}_E(\alpha,\delta)\underset{\simeq}{E}^{\bullet}$ the second base vector $\underset{\sim}{E}_{2''}$ lies in the equatorial plane, but in $\underset{\simeq}{E}'$ the vector $\underset{\sim}{E}_{2'}$ lies in the horizontal plane (east). Of course $\underset{\simeq}{E}'$ and $\underset{\simeq}{E}''$ have the common third base vector, but they differ by a rotation round this vector with an angle ξ :

$$\underset{\simeq}{E}'' = \underline{R}_3(\xi)\underset{\simeq}{E}' \quad . \tag{1-12}$$

A serial connection of several transformations is called a diagram in the algebra. If the transformations which have to be reversibly unequivocal, form a closed circle, this is called a commutative diagram. Then one is able to express a transformation by means of the others. Such a commutative diagram is presented in Fig. 1 by the lines which are thickly marked. For example the transformation $\underset{\simeq}{E}^* \rightarrow \underset{\simeq}{E}'$ can be expressed by

$$\underset{\simeq}{E}' = \underline{R}_E(T,B)\underline{R}_3(\Sigma)\underset{\simeq}{E}^* \tag{1-13}$$

or

$$\underset{\simeq}{E}' = \underline{R}_3^T(\xi)\underline{R}_E(\alpha,\delta)\underline{R}_3^T(\theta_{Gr})\underline{R}_E^T(\Lambda,\Phi)\underset{\simeq}{E}^* \quad . \tag{1-14}$$

As the representation of the triad $\underset{\simeq}{E}'$ with regard to the triad $\underset{\simeq}{E}^*$ is unequivocal, it follows that:

$$\underline{R}_E(T,B)\underline{R}_3(\Sigma)=\underline{R}_3^T(\xi)\underline{R}_E(\alpha,\delta)\underline{R}_3^T(\theta_{Gr})\underline{R}_E^T(\Lambda,\Phi) \qquad (1\text{-}15)$$

This equation reads with the right-hand side written in full length

$$\underline{R}_E(T,B)\underline{R}_3(\Sigma)=\underline{R}_3(-\xi)\underline{R}_2(90°-\delta)\underline{R}_3(\alpha)\underline{R}_3(-\theta_{Gr})\underline{R}_3(-\Lambda)\underline{R}_2(\Phi-90°) \qquad (1\text{-}16)$$

and can again be reduced to

$$\underline{R}_E(T+\Sigma,B)=\underline{R}_e^T(\xi,\delta-90°)\underline{R}_E^T(\theta_{Gr}+\Lambda-\alpha,\Phi) \qquad (1\text{-}17)$$

These are the desired *fundamental relations* between the parameters appearing in geodetic astronomy.

2. The observation equations of geodetic astronomy

The fundamental equation (1-17) consists as a matrix equation of nine separate equations, of which only three are independent of each other because of the property of orthonormality of the rotation matrices. These three independent equations represent *condition equations with unknowns* for every star (if T,B and θ_{Gr} are measured at one instant).

The matrices on the left and right-hand side of equation (1-17) read as follows when they are multiplied respectively:

$$\begin{bmatrix} \cos(\Sigma+T)\sin B & \sin(\Sigma+T)\sin B & -\cos B \\ -\sin(\Sigma+T) & \cos(\Sigma+T) & 0 \\ \cos(\Sigma+T)\cos B & \sin(\Sigma+T)\cos B & \sin B \end{bmatrix}$$

and

Column 1:

$\sin\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\sin\delta\cos\xi-\sin\Phi\sin(\theta_{Gr}+\Lambda-\alpha)\sin\xi+\cos\Phi\cos\delta\cos\xi$

$\sin\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\sin\delta\sin\xi+\sin\Phi\sin(\theta_{Gr}+\Lambda-\alpha)\cos\xi+\cos\Phi\cos\delta\sin\xi$

$\sin\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\cos\delta-\cos\Phi\sin\delta$

Column 2:

$-\sin(\theta_{Gr}+\Lambda-\alpha)\sin\delta\cos\xi-\cos(\theta_{Gr}+\Lambda-\alpha)\sin\xi$

$-\sin(\theta_{Gr}+\Lambda-\alpha)\sin\delta\sin\xi+\cos(\theta_{Gr}+\Lambda-\alpha)\cos\xi$

$-\sin(\theta_{Gr}+\Lambda-\alpha)\cos\delta$

Column 3:

$\cos\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\sin\delta\cos\xi-\cos\Phi\sin(\theta_{Gr}+\Lambda-\alpha)\sin\xi-\sin\Phi\cos\delta\cos\xi$

$\cos\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\sin\delta\sin\xi+\cos\Phi\sin(\theta_{Gr}+\Lambda-\alpha)\cos\xi-\sin\Phi\cos\delta\sin\xi$

$\cos\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\cos\delta+\sin\Phi\sin\delta$

In the right-hand matrix the elements in the third row are the shortest and at the same time the only ones which do not contain the angle ξ. Therefore, it is the obvious thing to do to select two equations from this row as independent equations. As a third equation one could take an element from another row of the matrices whereby the angle ξ , in which one is not actually interested, would indeed appear as an additional unknown. So one, therefore, dispenses with such an equation and there remain only two independent equations for one complete observation (T,B and θ_{Gr}) with the three unknowns Λ, Φ,Σ:

$$\sin B = \cos\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\cos\delta+\sin\Phi\sin\delta \tag{2-1}$$

$$\sin(\Sigma+T)\cos B = -\sin(\theta_{Gr}+\Lambda-\alpha)\cos\delta \tag{2-2}$$

$$\cos(\Sigma+T)\cos B = \sin\Phi\cos(\theta_{Gr}+\Lambda-\alpha)\cos\delta-\cos\Phi\sin\delta \tag{2-3}$$

Equations (2-1) and (2-2) are independent of each other, equation (2-3) is dependent on them both. It will be used later only for the determination of approximate values. The appearing variables be summarized once more:

Λ	astronomical longitude
Φ	astronomical latitude
α	right ascension of the star
δ	declination of the star
$h = \theta_{Gr} + \Lambda - \alpha$	hour angle
Σ	orientation unknown of the instrument (theodolite)
T	horizontal direction; observed
$\Rightarrow A = \Sigma + T$	azimuth
$B = 90° - z$	vertical direction, angle between horizon and star; observed
θ_{Gr}	Greenwich apparent sidereal time; observed

2.1 Linearization and matrix representation

The equations (2-1) and (2-2) are linearized by Taylor series where we build in *prior stochastic information* of the star positions α, δ.

Observations:

The horizontal direction T, the vertical direction B and the coordinate clock time τ are *observed*. In order to derive linearized condition equations from (2-1) and (2-2) horizontal and vertical directions are decomposed into

$$\boxed{T = \hat{t} + \Delta t \ , \quad B = \hat{b} + \Delta b \ .} \tag{2-4}$$

The coordinate clock time τ has to be related to the Greenwich apparent sidereal time angle θ_{Gr}, e.g. by the series

$$\theta_{Gr}(\tau) = \theta_{Gr}(\tau_o) + \dot{\theta}_{Gr}(\tau_o)(\tau - \tau_o) + o_2[(\tau - \tau_o)^2] \tag{2-5}$$

$$\tau = \tau_o + \dot{\theta}_{Gr}(\tau_o)\ [\theta_{Gr}(\tau) - \theta_{Gr}(\tau_o)] \ . \tag{2-6}$$

$\dot{\theta}_{Gr}$ may be identified with the earth rotation speed Ω.

Condition equations:

Ansatz: $\quad F[E\{Y\},\ X_1,\ X_2] = 0 \qquad$ (2-7)

The nonlinear vectorial condition equation $F[E\{Y\},\ X_1,\ X_2]$ contains the observation vector Y, (*stochastic* according to the theory of measurements), the *fixed* unknown vector X_1, but with *dispersive stochastic prior information* being available, and the *fixed* unknown vector with any stochastic prior information.

1st Taylor series

At a *fixed point* y, x_1, x_2 of approximation we are linearizing the system of nonlinear condition equations (2-7).

$$F[E\{Y\},\ X_1,\ X_2] = F(y,x_1,x_2) + F'(y,x_1,x_2)\ [E\{Y\} - y] + F'_{x_1}(y,x_1,x_2)(X_1 - x_1) + F'_{x_2}(y,x_1,x_2)(X_2 - x_2) + o_2 \tag{2-8}$$

o_2 indicates the neglected terms of higher order.

2nd Taylor series

At a *stochastic point* $\hat{y}$, $\hat{x}_1$ of approximation we are linearizing the system of nonlinear condition equations (2-7).

$$F(\hat{y},\hat{x}_1,x_2) = F(y,x_1,x_2) + F'_y(y,x_1,x_2)(\hat{y}-y) + \\ + F'_{x_1}(y,x_1,x_2)(\hat{x}_1-x) + \hat{o}_2 \qquad (2\text{-}9)$$

$$\left.\begin{matrix}(2\text{-}8)\\(2\text{-}9)\end{matrix}\right] \Longrightarrow$$

$$\boxed{\begin{aligned}F[E\{Y\},X_1,X_2] - F(\hat{y},\hat{x}_1,x_2) &= F'_y(y,x_1,x_2)[E\{Y\} - \hat{y}] \\ &+ F'_{x_1}(y,x_1,x_2)(X_1-\hat{x}_1) + F'_{x_2}(y,x_1,x_2)(X_2-x_2)\end{aligned}} \qquad (2\text{-}10)$$

Now we have been led to a system of *linearized condition equations* where $Y - \hat{y} =: \Delta y$ and $X_1 - \hat{x}_1 =: \Delta x_1$ are *stochastic correction vectors* while $X_2 - x_2 =: \Delta x_2$ contains the fixed or *non-stochastic correction vector*. The next step is to write down the *observational equation* for $\hat{y}$ and the *pseudo-observational equation* for $\hat{x}_1$, the vector of stochastic information.

$$\boxed{\begin{aligned}&E\{Y\} \ , \quad D\{Y\} = Q\sigma^2 \ , \quad \text{rk } Q = n \\ &E\{\hat{x}_1\} = X_1 - b_{x_1} \ , \quad D\{\hat{x}_1\} = V_1\sigma^2 \ , \quad \text{rk } V_1 \leq m_1 \ .\end{aligned}} \qquad (2\text{-}11)$$

Here the *pseudo-observational equation* is generated by prior informative data of the *star positions*, parameterized by $\{\alpha,\delta\}$, namely by

$$E\left\{\begin{bmatrix}\hat{\alpha}\\ \hat{\delta}\end{bmatrix}\right\} = \begin{bmatrix}\alpha\\ \delta\end{bmatrix} - \begin{bmatrix}b_\alpha\\ b_\delta\end{bmatrix} \ , \quad D\left\{\begin{bmatrix}\hat{\alpha}\\ \hat{\delta}\end{bmatrix}\right\} = V_1\sigma^2 \ , \qquad (2\text{-}12)$$

$$\hat{x}_1 =: [\hat{\alpha},\hat{\delta}]^T \ . \qquad (2\text{-}13)$$

$$\begin{aligned}&\sin B = \sin(\hat{b}+\Delta b) \doteq \sin\hat{b} + \Delta b \cos\hat{b} \\ &\cos\Phi = \cos(\Phi_o+\delta\Phi) \doteq \cos\Phi_o - \delta\Phi \sin\Phi_o \\ &\cos(\theta_{Gr}+\Lambda-\alpha) = \cos({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha}+\delta\theta_{Gr}+\delta\Lambda-\delta\hat{\alpha}) \doteq \\ &\doteq \cos({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha}) - (\delta\theta_{Gr}+\delta\Lambda-\delta\hat{\alpha})\sin({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha}) \\ &\cos\delta = \cos(\delta+\delta\hat{\delta}) \doteq \cos\hat{\delta} - \delta\hat{\delta} \sin\hat{\delta} \\ &\sin\Phi = \sin(\Phi_o+\delta\Phi) \doteq \sin\Phi_o + \delta\Phi \cos\Phi_o\end{aligned} \qquad (2\text{-}14)$$

(2-1), (2-2), (2-14) $\longrightarrow$

$$
\boxed{
\begin{aligned}
\sin\hat{b} + \cos\hat{b}\,\Delta b \doteq{} & \cos\Phi_o \cos({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})\cos\hat{\delta} + \sin\Phi_o\,\sin\hat{\delta} - \\
& - \cos\Phi_o\,\sin({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})(\delta\theta_{Gr}+\delta\Lambda) \\
& + [-\sin\Phi_o\,\cos({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha}) + \cos\Phi_o\,\sin\hat{\delta}]\,\delta\Phi \\
& + \cos\Phi_o\,\sin({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})\,\hat{\delta\alpha} + \\
& + \sin\Phi_o(\cos\hat{\delta})\,\delta\hat{\delta}\ ;
\end{aligned}
}
\tag{2-15}
$$

$$
\boxed{
\begin{aligned}
& \sin(\Sigma_o+\hat{t})\cos\hat{b} + \cos(\Sigma_o+\hat{t})\cos\hat{b}(\delta\Sigma+\Delta t) - \\
& -\sin(\Sigma_o+\hat{t})\sin\hat{b}\,\Delta b = \\
& -\sin({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})\cos\hat{\delta} - \cos({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})\cos\hat{\delta}(\delta\theta_{Gr}+\delta\Lambda) \\
& +\cos({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})\delta\hat{\alpha} + \sin({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})(\sin\hat{\delta})\,\delta\hat{\delta}
\end{aligned}
}
\tag{2-16}
$$

In extension of the linearized condition equations (2-4), (2-5), respectively being introduced in the *First Technical Report*, page 13, here we are led to the additional terms proportional $\delta\hat{\alpha}$, $\delta\hat{\delta}$, respectively. They account for the *uncertainties of the star positions*. Once we assume zero $F(Y,X_1,X_2) - F(\hat{y},\hat{x}_1,x_2) = 0$, a result which can be achieved by a proper gauging of the approximate values, we finally introduce the linearized equations

$$
\boxed{
\begin{aligned}
& -\cos\hat{b}\,\Delta b - \cos\Phi_o\,\sin({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})(\delta\theta_{Gr}+\delta\Lambda) + \\
& + [-\sin\Phi_o\,\cos({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha}) + \cos\Phi_o\sin\hat{\delta}]\,\delta\Phi + \\
& + \cos\Phi_o\sin({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})\delta\hat{\alpha} + \sin\Phi_o(\cos\hat{\delta})\,\delta\hat{\delta} = 0
\end{aligned}
}
\tag{2-17}
$$

$$
\boxed{
\begin{aligned}
& -\cos(\Sigma_o+t)\cos\hat{b}(\delta\Sigma+\Delta t) + \sin(\Sigma_o+\hat{t})\sin\hat{b}\,\Delta b - \\
& -\cos({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})\cos\hat{\delta}(\delta\theta_{Gr}+\delta\Lambda) + \\
& +\cos({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})\,\delta\hat{\delta} + \sin({}_o\theta_{Gr}+\Lambda_o-\hat{\alpha})(\sin\hat{\delta})\,\delta\hat{\delta} = 0
\end{aligned}
}
\tag{2-18}
$$

From these equations we obtain the proper condition equations with respect to the estimation / prediction problem when we introduce the *vectors of inconsistency*, e_y and e_{x_1}, respectively.

$$
\boxed{Y = E\{Y\} + e_y\ ,\quad \hat{x}_1 = E\{\hat{x}_1\} + e_{x_1}}
\tag{2-19}
$$

$$
F'_y\Delta y + F'_{x_1}\Delta x_1 + F'_{x_2}\Delta x_2 - F'_y e_y = 0
\tag{2-20}
$$

$$
F'_y\Delta\hat{y} + F'_{x_1}b_{x_1} + F'_{x_2}\Delta x_2 - F'_{x_1}e_{x_1} - F'_y e_y = 0
\tag{2-21}
$$

$$\boxed{A_1 b_1 + A_2 \Delta x_2 + B\Delta y = A_1 e_1 + Be_y} \tag{2-22}$$

A_1, A_2 and B are the coefficient matrices for all observations taken to various stars. The structure of the coefficient matrices can be read from the two equations (2-17) and (2-18). Note that not only Λ, Φ, Σ or $\delta\Lambda$, $\delta\Phi$, $\delta\Sigma$ are *unknowns*, but as well α, δ or $\hat{\delta\alpha}$, $\hat{\delta\delta}$, the corrections to star positions. Their uncertainty will play an essential part in the final computation of the variance-covariance matrix of Λ, Φ, Σ!

In order to solve the linearized mixed model condition equations (2-22) we apply the *generalized method of least-squares*

$$\begin{aligned}
&L\{e_y, e_{x_1}, \Delta x_2, \lambda\} = \\
&= \frac{1}{2}[e_y^T, e_{x_1}^T]\left\{\begin{bmatrix} Q & 0 \\ 0 & V_1 \end{bmatrix} + \begin{bmatrix} A_1 \\ I_m \end{bmatrix}[A_1^T, I_m]\right\}^{-1}\begin{bmatrix} e_y \\ e_{x_1} \end{bmatrix} + \\
&+ \lambda^T(A_1 b_1 + A_2\Delta x_2 + B\Delta y - A_1 e_{x_1} - Be_y = \\
&= \frac{1}{2}[e_y^T, e_{x_1}^T]\begin{bmatrix} Q + A_1 A_1^T, & A_1 \\ A_1^T & , I_m + V_1 \end{bmatrix}^{-1}\begin{bmatrix} e_y \\ e_{x_1} \end{bmatrix} + \\
&+ \lambda^T(A_1 b_1 + A_2\Delta x_2 + B\Delta y - A_1 e_{x_1} - Be_y) = \min_{(e_y, e_{x_1}, \Delta x_2, \lambda)} .
\end{aligned} \tag{2-23}$$

The normal equations and their solution will be given in the *Final Report*. There an extension of the results, *First Technical Report*, pages 20-29, will be presented.